Lecture 21: Spectral Sequence (II)

Spectral sequence for filtered cochain complex

Definition

A filtered cochain complex is a cochain complex $(C^\bullet, d)$ with a (descending) filtration

$$\cdots \supset F_p C^i \supset F_{p+1} C^i \supset \cdots$$

of each C^i such that the differential preserves the filtration

$$d(F_p C^i) \subset F_p C^{i+1}.$$

In other words, we have a decreasing sequence of subcomplexes

$$F_p C^\bullet \subset C^\bullet.$$

The associated graded complex is

$$\mathrm{Gr}_p^F C^\bullet = F_p C^\bullet / F_{p+1} C^\bullet.$$

The convention for a special sequence in this case is

- an R-module $E_r^{p,q}$ for any $p, q \in \mathbb{Z}$ and $r \geq 0$;
- a differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$ and

$$E_{r+1} = \mathrm{H}(E_r, d_r).$$

Theorem

There is an associated spectral sequence for any filtered cochain complex $(C^\bullet, d, F_\bullet)$ where

$$E_r^{p,q} = \frac{\{x \in F_p C^{p+q} | dx \in F_{p+r} C^{p+q+1}\}}{F_{p+1} C^{p+q} + dF_{p-r+1} C^{p+q-1}}.$$

and

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad x \to dx.$$

The E_1-page of the spectral sequence is

$$E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}_p^F C^\bullet).$$

If the filtration of C^i is bounded for each i, then the spectral sequence converges and

$$E_\infty^{p,q} = \mathrm{Gr}_p \mathrm{H}^{p+q}(C^\bullet).$$

Double complex

Let us come back to the double complex example

$$K = \bigoplus_{p,q \geq 0} K^{p,q}$$

which is equipped with two differentials

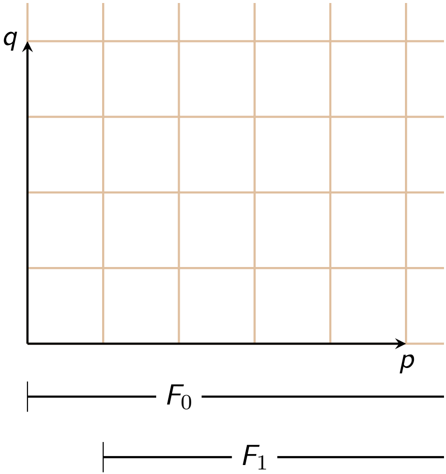
$$\begin{cases} \delta_1 : K^{p,q} \to K^{p,q+1} \\ \delta_2 : K^{p,q} \to K^{p+1,q} \end{cases}$$

We want to compute the cohomology of the total complex

$$\mathrm{H}^\bullet(\mathrm{Tot}^\bullet(K), D), \quad D = \delta_1 + \delta_2.$$

Let us define a descending filtration on K by

$$F_pK = \bigoplus_{m \geq p, n \geq 0} K^{m,n}.$$



This induces a descending filtration on $\mathrm{Tor}^{\bullet}(K)$ by

$$F_p \mathrm{Tor}^{\bullet}(K) := \mathrm{Tor}^{\bullet}(F_p K)$$

whose graded associated complex is

$$\mathrm{Gr}_p \mathrm{Tor}^{\bullet}(K) = \bigoplus_{q \geq 0} K^{p,q}, \quad \text{differential} = \delta_1.$$

The E_1 page of the spectral sequence is

$$E_1^{p,q} = \mathrm{H}_{\delta_1}^{p,q}(K), \quad d_1 = \delta_2.$$

The E_2 page of the spectral sequence is

$$E_2^{p,q} = \mathrm{H}_{\delta_2}^{p,q}\,\mathrm{H}_{\delta_1}(K).$$

An element of $E_r^{p,q}$ is represented by an $x_0 \in K^{p,q}$ that can be extended to a chain

$$x = x_0 + x_1 + \cdots + x_{r-1}, \quad x_i \in K^{p+i,q-i}$$

such that

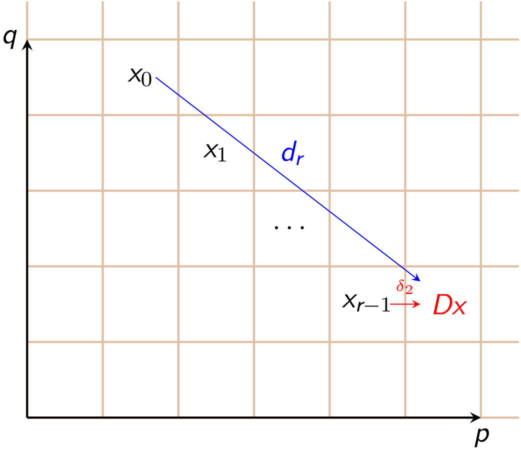
$$Dx \in K^{p+r+1,q-r}.$$

In other words, we can solve the following equations up to x_{r-1}

$$\begin{cases} \delta_1 x_0 = 0 \\ \delta_2 x_0 = -\delta_1 x_1 \\ \delta_2 x_1 = -\delta_1 x_2 \\ \quad \vdots \\ \delta_2 x_{r-2} = -\delta_1 x_{r-1}. \end{cases}$$

The corresponding differential for the E_r-page is

$$d_r[x_0] = [Dx] = [\delta_2 x_{r-1}].$$



Cellular chain complex revisited

Let X be a CW complex with cellular structure

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} \subset \cdots .$$

We define an ascending filtration on the singular chain complex by

$$F_p S_\bullet(X) = S_\bullet(X^{(p)}).$$

The E^0-page is

$$E^0_{p,q} = \mathrm{Gr}_p(S_{p+q}(X)) = \frac{S_{p+q}(X^{(p)})}{S_{p+q}(X^{(p-1)})} = S_{p+q}(X^{(p)}, X^{(p-1)}).$$

Therefore the E^1-page computes the relative homology

$$E^1_{p,q} = \mathrm{H}_{p+q}(X^{(p)}, X^{(p-1)}) = \begin{cases} C^{cell}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

which gives precisely the cellular chains.

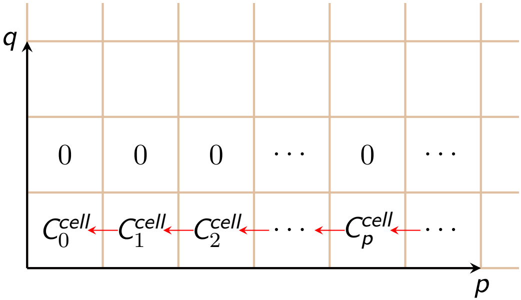


图: E^1-page

The differential ∂_1 coincides with the cellular differential

$$\partial : C_p^{cell}(X) \to C_{p-1}^{cell}(X).$$

Therefore the E^2-page is

$$E^2_{p,q} = \begin{cases} \mathrm{H}_p^{cell}(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

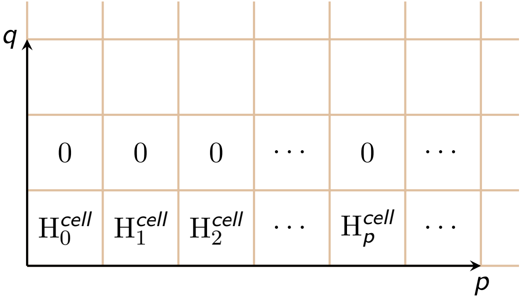


图: E^2-page

The shape of this E^2-page implies that

$$\partial_2 = \partial_3 = \cdots = 0, \quad \Longrightarrow E^2 = E^3 = \cdots = E^\infty.$$

This explains why cellular homology computes singular homology.

Leray-Serre spectral sequence

Let $\pi : E \to B$ be a Serre fibration with fiber F and base B.

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

Assume B is a simply-connected CW complex. Then there is the Leray-Serre spectral sequence with E^2-page

$$E^2_{p,q} = \mathrm{H}_p(B) \otimes \mathrm{H}_q(F)$$

that converges to $\mathrm{Gr}_p \, \mathrm{H}_{p+q}(E)$.

The idea of this spectral sequence is that we can filter the singular chain complex of E such that it favors for the computation of singular homology along the fiber first. Explicitly, we can use

$$B^{(0)} \subset B^{(1)} \subset \cdots \subset B^{(n)} \subset \cdots$$

to obtain a filtration of topological spaces for E

$$E^{(0)} \subset E^{(1)} \subset \cdots \subset E^{(n)} \subset \cdots$$

where $E^{(n)}$ is given the pull-back

$$\begin{array}{ccc} E^{(n)} & \longrightarrow & E \\ \downarrow & & \downarrow \\ B^{(n)} & \longrightarrow & B \end{array}$$

Example

Consider the fibration ($n \geq 2$)

$$\begin{array}{ccc} \Omega S^n & \longrightarrow & P\Omega^n \\ & & \downarrow \\ & & S^n \end{array}$$

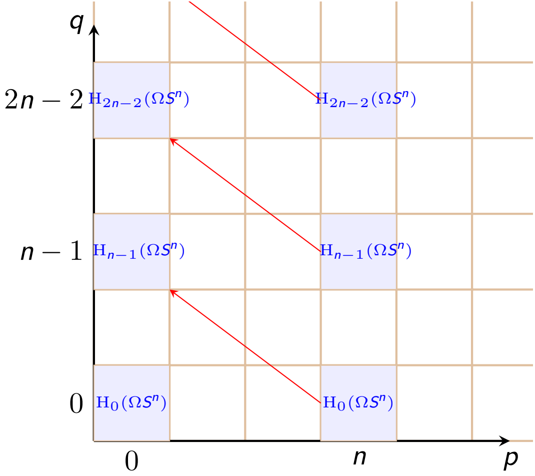
Here $P\Omega^n$ is the based path space of S^n. We have

$$\mathrm{H}_p(S^n) = \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & p \neq 0, n \end{cases} \qquad \mathrm{H}_k(P\Omega^n) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0 \end{cases}$$

To arrive at $\mathrm{H}_\bullet(P\Omega^n)$, the Leray-Serre spectral sequence must have

$$E^2 = E^3 = \cdots = E^n$$

where the only non-zero terms are in the shaded locations below.



Furthermore, the maps

$$d_n : \mathrm{H}_{(n-1)k}(\Omega S^n) \to \mathrm{H}_{(n-1)(k+1)}(\Omega S^n), \quad k \geq 0$$

must be isomorphisms in order to have $E^\infty = \mathrm{Gr}\,\mathrm{H}_\bullet(P\Omega^n) = \mathbb{Z}$.

We conclude that

$$\mathrm{H}_i(\Omega S^n) = \begin{cases} \mathbb{Z} & i = k(n-1) \\ 0 & \text{otherwise} \end{cases}$$

Example

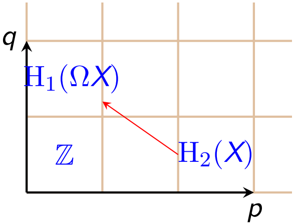
We illustrate Serre's approach to Hurewicz Theorem via spectral sequence.

Assume we have established Hurewicz Theorem for the $n = 1$ case $\pi_1 \to \mathrm{H}_1$. We prove by induction for the $n \geq 2$ case.

Let $n \geq 2$ and X be a $(n-1)$-connected CW complex. Consider the fibration

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ & & \downarrow \\ & & X \end{array}$$

The E^2-page of the Leray-Serre spectral sequence is



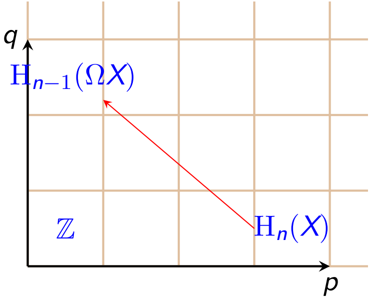
Since PX is contractible, the map

$$\mathrm{H}_2(X) \to \mathrm{H}_1(\Omega_X)$$

must be an isomorphism. This shows

$$\mathrm{H}_2(X) = \mathrm{H}_1(\Omega_X) = \pi_1(\Omega_X) = \pi_2(X) \quad (= 0 \text{ if } n > 2).$$

We can iterate this until we arrive at the E^n-page



Again by the contractibility of PX, ∂_r must induce an isomorphism

$$\mathrm{H}_n(X) = \mathrm{H}_{n-1}(\Omega X) \overset{\text{induction}}{=} \pi_{n-1}(\Omega X) = \pi_n(X).$$

This is the Hurewicz isomorphism.